Math 217 Fall 2025
Quiz 23 – Solutions

Dr. Samir Donmazov

1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_r\} \subset \mathbb{R}^n$ is *orthonormal* provided that ...

   Solution: Each vector has unit length and distinct vectors are orthogonal, i.e.

   $$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

   (b) Two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are *orthogonal* (perpendicular) provided that ...

   Solution: Their dot product is zero: $\vec{v} \cdot \vec{w} = 0$.

   (c) The *length* of a vector $\vec{v} \in \mathbb{R}^n$ is ...

   Solution: Its Euclidean norm: $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$, where $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

   (d) Given any set $S \subseteq \mathbb{R}^n$, the *orthogonal complement* of S is ...

   Solution: The set

   $$S^\perp = \{\vec{w} \in \mathbb{R}^n : \ \vec{w} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in S\},$$

   which is a subspace of $\mathbb{R}^n$.

2. Suppose V is a vector space and $v_0 \in V$. Suppose also that $v_1, \ldots, v_m$ are linearly independent vectors in V. Show that

$$v_0 \in \mathrm{Span}(v_1, \ldots, v_m) \quad \Longleftrightarrow \quad \{v_0, v_1, \ldots, v_m\} \text{ is linearly dependent.}$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: ($\Rightarrow$) If $v_0 = \sum_{i=1}^m c_i v_i$ for some scalars c_i, then

$$v_0 - \sum_{i=1}^m c_i v_i = 0$$

is a nontrivial relation among $v_0, v_1, \ldots, v_m$, so they are linearly dependent.

($\Leftarrow$) If $v_0, v_1, \ldots, v_m$ are linearly dependent, there exist scalars $a_0, a_1, \ldots, a_m$, not all zero, with

$$a_0 v_0 + \sum_{i=1}^m a_i v_i = 0.$$

If $a_0 = 0$, then $\sum_{i=1}^m a_i v_i = 0$ is a nontrivial relation among $v_1, \ldots, v_m$, contradicting their linear independence. Hence $a_0 \neq 0$, and

$$v_0 = -\sum_{i=1}^m \frac{a_i}{a_0} v_i \in \mathrm{Span}(v_1, \ldots, v_m).$$

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) We have $W \cap W^\perp = \{\vec{0}\}$.

Solution: FALSE in general if W is not assumed to be a subspace (so it may not contain $\vec{0}$). For example, in $\mathbb{R}^2$ let $W = \{\vec{e}_1\}$. Then $W^\perp = \{(0,t) : t \in \mathbb{R}\}$ and $W \cap W^\perp = \varnothing \neq \{\vec{0}\}$.

Remark. If W *is* a subspace, then W contains $\vec{0}$ and the usual argument shows $W \cap W^\perp = \{\vec{0}\}$: if $\vec{x} \in W \cap W^\perp$ then $\vec{x} \cdot \vec{x} = 0$, hence $\vec{x} = \vec{0}$.

(b) Each $\vec{v} \in \mathbb{R}^n$ decomposes *uniquely* as $\vec{v} = \vec{v}^\parallel + \vec{v}^\perp$ where $\vec{v}^\parallel \in W$ and $\vec{v}^\perp \in W^\perp$.

Solution: FALSE in general if W is not a subspace. Again take $W = \{\vec{e}_1\} \subset \mathbb{R}^2$. Then $W^\perp = \{(0,t)\}$. For $\vec{v} = 2\vec{e}_1$, a decomposition $\vec{v} = \vec{v}^\parallel + \vec{v}^\perp$ with $\vec{v}^\parallel \in W$ forces $\vec{v}^\parallel = \vec{e}_1$, but then $\vec{v}^\perp = \vec{e}_1 \notin W^\perp$. Thus no such decomposition exists.

Remark. If W *is* a subspace, both existence and uniqueness hold (orthogonal projection): with an orthonormal basis $\{w_1, \ldots, w_k\}$ for W,

$$\vec{v}^\parallel = \sum_{j=1}^k (\vec{v} \cdot w_j) w_j \in W, \qquad \vec{v}^\perp = \vec{v} - \vec{v}^\parallel \in W^\perp,$$

and uniqueness follows from $W \cap W^\perp = \{\vec{0}\}$.